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CROSSED PRODUCTS OF II_1-SUBFACTORS BY STRONGLY OUTER ACTIONS

CARL WINSLØW

ABSTRACT. We study the crossed product $A \rtimes G \supseteq B \rtimes G$ of an inclusion $A \supseteq B$ of type II_1-factors by a discrete strongly outer action G. In particular, we find conditions under which the strong amenability of $A \supseteq B$ implies that of $A \rtimes G \supseteq B \rtimes G$, and vice versa.

1. INTRODUCTION

Strongly amenable inclusions of factors of type II_1 were introduced and classified by Popa in [11] (though the possible values of the involved invariant are not in general known). Strong amenability is similar to injectivity in the case of single factors, and its equivalence with the generating property (i.e., the existence of an approximating tunnel for the inclusion) is an analogue to Connes' fundamental result that injective factors are hyperfinite [3].

In another direction, Choda and Kosaki [2] and Popa [12] have introduced an outerness condition for actions of discrete groups on a subfactor. Actions satisfying this property are called *strongly outer* (cf. §2). In the case of strongly amenable II_1-subfactors and amenable groups, such actions were classified up to cocycle conjugacy by Popa [12]. The case of infinite subfactors was handled in [13], where the so-called strongly free actions were defined and classified.

Since strongly amenable subfactors can be classified, it is of course important to determine when a given subfactor is strongly amenable. Various criteria were presented in [11], and in this paper we shall apply some of them to the case of crossed products of II_1-subfactors by a strongly outer action of a discrete amenable group. We show that whenever the crossed product of a II_1-subfactor by a strongly outer action of a discrete group is strongly amenable, then the II_1-subfactor we started with is actually strongly amenable. The converse implication is shown only for finite groups, though a strategy for the general case is indicated. The point of our approach is to explicitly construct a tunnel for the crossed product inclusion in terms of a given tunnel for the subfactor we start with.

The main references for this paper are [5] and [11], where the reader can find the basic notions and results used here.

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2. NOTATION

Let $A \supseteq B$ be a finite index inclusion of hyperfinite factors of type II_1. We fix a choice of the tunnel

$$B_{-1} = A \supseteq B_0 = B \supseteq B_1 \supseteq B_2 \supseteq \cdots$$

for $A \supseteq B$, together with Jones projections $e_{-j} \in B_{j-1} \cap B'_{j+1}$, $j \geq 0$.

Let G be a discrete group, let $\alpha: G \to \mathrm{Aut}(A, B)$ be an action of G on $A \supseteq B$, and put

$$\widetilde{A} = A \rtimes_\alpha G, \qquad \widetilde{B} = B \rtimes_\alpha G.$$

When it causes no confusion, we identify $A \supseteq B$ with its image in $\widetilde{A} \supseteq \widetilde{B}$.

For X a subalgebra of $\widetilde{A}$, E_X denotes the trace-preserving conditional expectation onto X. Also let $\|\cdot\|_2$ denote the weak norm induced by the tracial state on $\widetilde{A}$.

We assume that α is strongly outer in the sense of [2, 2.1], or, equivalently (by [13, 3.1]), that α is properly outer as defined in [12, 1.5], i.e.,

$$\widetilde{A} \cap B'_k = A \cap B'_k, \qquad k \geq 0.$$

In particular, α is outer on both A and B, so $\widetilde{A} \supseteq \widetilde{B}$ is an inclusion of II_1-factors. Also $[\widetilde{A} : \widetilde{B}] = [A : B] < \infty$.

We finally assume that $\widetilde{A}$ and $\widetilde{B}$ are hyperfinite, which is equivalent to the assumption that G is amenable (see [3, 6.8; 6, p. 96]).

3. EXTREMALITY

Recall from [11, 1.2.5] that $A \supseteq B$ is called *extremal* if $E_{A \cap B'}(e_0) \in \mathbb{C}1$. If $[A : B]_0$ denotes the minimal index [4] of $A \supseteq B$, then $A \supseteq B$ is extremal if and only if $[A : B] = [A : B]_0$.

3.1. **Lemma.** *If $A \supseteq B$ is extremal, then so is $\widetilde{A} \supseteq \widetilde{B}$.*

Proof. Since the trace-preserving conditional expectation of $\widetilde{A}$ onto $\widetilde{B}$ is an extension of the one of A onto B, we get by definition that e_0 is also a Jones projection for $\widetilde{A} \supseteq \widetilde{B}$. By the strong outerness (in fact, less suffices), we have

$$\widetilde{A} \cap \widetilde{B}' \subseteq \widetilde{A} \cap B' = A \cap B'.$$

Hence

$$E_{\widetilde{A} \cap \widetilde{B}'}(e_0) = E_{\widetilde{A} \cap \widetilde{B}'} E_{A \cap B'}(e_0) \in \mathbb{C}1. \quad \square$$

We mention that the converse of the above lemma is also true and even in a more general setting. Another easy observation of this type is the following

3.2. **Lemma.** *Assume that the group G is finite and acts outerly on $A \supseteq B$, i.e., the action on A and its restriction to B are outer actions in the usual sense. Then $A^G \supseteq B^G$ is an inclusion of* II_1*-factors which is extremal if and only if $A \supseteq B$ is extremal.*

Proof. By [1, II.3] we have $A \cap (A^G)' = \mathbb{C}1$ and $B \cap (B^G)' = \mathbb{C}1$. In particular, the fixed-point algebras are factors and the inclusions $A \supseteq A^G$ and $B \supseteq B^G$ are irreducible, hence extremal. So, by [5, 2.3.3], we have $[A : A^G]_0 = [B : B^G]_0 = |G|$. Since the minimal index is multiplicative (see [9.2.2; 11, 1.2.5]), we get

$$[A : B]_0 = \frac{[A : B^G]_0}{[B : B^G]_0} = \frac{[A : A^G]_0 [A^G : B^G]_0}{[B : B^G]_0} = [A^G : B^G]_0 ,$$

and, by the same computation for the Jones index, $[A : B] = [A^G : B^G]$. □

4. The Loi part of an action

In order to construct an appropriate tunnel for $\widetilde{A} \supseteq \widetilde{B}$, we need the following generalization of Loi's construction in [7, §2]; a related result was obtained in [12, 3.4].

4.1. Lemma. *For each $k \geq 0$, there is an α-cocycle u^k such that, when we define $\alpha^{k+1} = \mathrm{Ad}(u^k)\alpha$, we have*

$$\alpha_g^{k+1}(e_{-j}) = e_{-j}, \qquad j \leq k, \ g \in G.$$

Moreover, these cocycles may be chosen as products

$$u^k = v^k \cdots v^0 \qquad (k \geq 0)$$

where $v^0 = u^0$ and v^j is an α^j-cocycle with unitaries coming from B_j for all $j \geq 1$.

Proof. By [7] we have unitaries $(z_g)_{g \in G} \subseteq B$ such that $\mathrm{Ad}(z_g)\alpha_g(e_0) = e_0$ for each $g \in G$. Put

$$\beta_g = \mathrm{Ad}(z_g)\alpha_g , \qquad g \in G,$$
$$v_{g,h} = z_g \alpha_g(z_h) z_{gh}^* , \qquad g, h \in G.$$

As $\beta_g(e_0) = e_0$ we have

$$\beta_g(B_1) = \beta_g(B \cap \{e_0\}') = B_1 , \qquad g \in G.$$

Since α and hence β is strongly outer, $\beta_g|_{B_1} \notin \mathrm{Int}(B_1)$ for $f \in G \backslash \{1\}$. Also, for $g, h \in G$,

$$v_{g,h} e_0 v_{g,h}^* = \mathrm{Ad}(v_{g,h})\beta_{gh}(e_0) = \beta_g \beta_h(e_0) = e_0$$

and hence $v_{g,h} \in B_1$. Thus $\{(\beta_g|_{B_1})_{g \in G}, (v_{g,h})_{g,h \in G}\}$ is a free cocycle crossed action of G on B_1 at which we can apply Ocneanu's 2-cocycle vanishing theorem [10, 1.1]. This produces unitaries $(w_g)_{g \in G}$ in B_1 such that $u_g^0 = w_g z_g$ ($g \in G$) defines an α-cocycle u^0 satisfying $\mathrm{Ad}(u_g^0)\alpha_g(e_0) = e_0$ for all $g \in G$.

Clearly this argument may be continued up to any k to prove our claim in general. □

4.2. Definition. When $A \supseteq B$ is strongly amenable, we can, by [11, 4.1.2], assume that the tunnel $A \supseteq B \supseteq B_1 \supseteq \cdots$ is a generating tunnel, i.e.,

$$\left(\bigvee_{k=0}^{\infty} A \cap B_k' \supseteq \bigvee_{k=0}^{\infty} B \cap B_k' \right) = (A \supseteq B).$$

Then the sequence $(\alpha^k|_{A \cap B_k'})_{k=0}^{\infty}$ defines an action on $A \supseteq B$ which we, in analogy with the case of single automorphisms [12, 1.2; 7, §5], denote by α^{st} and call the *standard* (or *Loi*) *part* of α.

Note that we can arrive at the Loi part of α using any choice of the unitaries from [7] (the z_g in the proof of (4.1)). The point of the above is that α^{st} can be constructed using cocycles, namely, as a kind of limit of the actions a^k.

5. A TUNNEL FOR THE CROSSED PRODUCT

Assume the inclusion $A \supseteq B$ is initially represented in $\mathscr{B}(H)$ for some Hilbert space H. Then, as usual, $\widetilde{A} \supseteq \widetilde{B}$ acts on $\widetilde{H} = \ell^2(G, H)$. Thus we have a canonical representation $\pi_\alpha: A \to \widetilde{A}$ given by

$$\pi_\alpha(x)\xi(g) = \alpha_{g^{-1}}(x)\xi(g), \qquad x \in A, \ g \in G, \ \xi \in \widetilde{H},$$

which together with the left regular representation of G in $\mathscr{B}(\widetilde{H})$ generates $\widetilde{A}$. The maps $\pi_{\alpha^k}: A \to A \rtimes_{\alpha^k} G$ are defined the same way.

Now, for each $k \geq 0$, we define unitaries $\tilde{u}_k$ and $\tilde{v}_k$ in $\mathscr{B}(\tilde{H})$ by

$$\tilde{u}_k\xi(g) = u^k_{g^{-1}}\xi(g), \qquad \xi \in \widetilde{H}, \ g \in G,$$
$$\tilde{v}_k\xi(g) = v^k_{g^{-1}}\xi(g), \qquad \xi \in \widetilde{H}, \ g \in G,$$

where u^k, v^k are as in the previous lemma. Then we have the following

5.1. **Proposition.** *With notation as above, we define*

$$\widetilde{B}_{-1} = \widetilde{A}; \qquad \widetilde{B}_0 = \widetilde{B}; \qquad \widetilde{B}_j = \tilde{u}^*_{j-1}(B_j \rtimes_{\alpha^j} G)\tilde{u}_{j-1}, \quad j \geq 1.$$

Then

$$\widetilde{A} \supseteq \widetilde{B} \supseteq \widetilde{B}_1 \supseteq \widetilde{B}_2 \supseteq \cdots$$

is a tunnel for $\widetilde{A} \supseteq \widetilde{B}$ *with corresponding Jones projections* $\pi_\alpha(e_{-j}) \in \widetilde{B}_{j-1} \cap \widetilde{B}'_{j+1}$ *for* $j \geq 0$. *Also,*

$$\widetilde{A} \cap \widetilde{B}'_j = \pi_\alpha(A) \cap \widetilde{B}'_j \quad \textit{for each } j \geq 0.$$

Proof. It follows from the definitions that

$$A \rtimes_{\alpha^k} G = \tilde{u}_{k-1}(A \rtimes_\alpha G)\tilde{u}^*_{k-1}, \qquad k \geq 0,$$

where in particular

$$\tilde{u}_{k-1}\pi_\alpha(x)\tilde{u}^*_{k-1} = \pi_{\alpha^k}(x), \qquad x \in A.$$

As in the proof of Lemma 3.1, we see that for each $k \geq 0$ the sequence of inclusions

$$A \rtimes_{\alpha^k} G \supseteq B \rtimes_{\alpha^k} G \supseteq B_1 \rtimes_{\alpha^k} G \supseteq \cdots \supseteq B_k \rtimes_{\alpha^k} G$$

is standard—i.e., is a series of downward basic constructions—with Jones projections $\pi_{\alpha^k}(e_{-j}) \in (B_{j-1} \rtimes_{\alpha^k} G) \cap (B_{j+1} \rtimes_{\alpha^k} G)'$ for $j \leq k-1$. Hence the sequence

$$\widetilde{A} \supseteq \widetilde{B} \supseteq \tilde{u}^*_{k-1}(B_1 \rtimes_{\alpha^k} G)\tilde{u}_{k-1} \supseteq \cdots \supseteq \tilde{u}^*_{k-1}(B_k \rtimes_{\alpha^k} G)\tilde{u}_{k-1}$$

is also standard, with Jones projections $\tilde{u}^*_{k-1}\pi_{\alpha^k}(e_{-j})\tilde{u}_{k-1} = \pi_\alpha(e_{-j})$. Here, by the above formulae, we have

$$\begin{aligned}\tilde{u}^*_{k-1}(B_j \rtimes_{\alpha^k} G)\tilde{u}_{k-1} &= \tilde{u}^*_{j-1}\tilde{v}^*_j \cdots \tilde{v}^*_{k-1}(B_j \rtimes_{\mathrm{Ad}(v^{k-1}\cdots v^j)\alpha^j} G)\tilde{v}_{k-1} \cdots \tilde{v}_j\tilde{u}_{j-1} \\ &= \tilde{u}^*_{j-1}(B_j \rtimes_{\alpha^j} G)\tilde{u}_{j-1} = \widetilde{B}_j, \qquad j \leq k-1,\end{aligned}$$

so, in particular, $\pi_\alpha(e_{-j}) \in \widetilde{B}_{j-1} \cap \widetilde{B}'_{j+1}$ for each j.

Finally, since each α^k is strongly outer, we get for all $k \geq 1$

$$\begin{aligned}\widetilde{A} \cap \widetilde{B}_k' &= \widetilde{A} \cap (\tilde{u}_{k-1}^*(B_k \rtimes_{\alpha^k} G)\tilde{u}_{k-1})' \\ &= \tilde{u}_{k-1}^*((A \rtimes_{\alpha^k} G) \cap (B_k \rtimes_{\alpha^k} G)')\tilde{u}_{k-1} \\ &= \tilde{u}_{k-1}^*(\pi_{\alpha^k}(A) \cap (B_k \rtimes_{\alpha^k} G)')\tilde{u}_{k-1} \\ &= \pi_\alpha(A) \cap \widetilde{B}_k'. \quad \square\end{aligned}$$

The previous result, as well as (4.1), also holds when $A \supseteq B$ is instead assumed to be of type II_∞, with the same proofs.

5.2. **Corollary.** *If $A \supseteq B$ is strongly amenable, then the standard part* [11, 1.4.1] *of $\widetilde{A} \supseteq \widetilde{B}$ is isomorphic to $A^{\alpha^{st}} \supseteq B^{\alpha^{st}}$.*

Proof. Since $A \supseteq B$ is strongly amenable, we can assume that the tunnel $A \supseteq B \supseteq B_1 \supseteq \cdots$ is a generating tunnel. Let $\lambda: G \to \widetilde{A}$ denote the canonical unitary representation in the crossed product. Then, by (5.1),

$$\begin{aligned}\widetilde{A} \cap \widetilde{B}_k' &= \pi_\alpha(A) \cap \widetilde{B}_k' \\ &= \tilde{u}_{k-1}^*(\pi_{\alpha^k}(A \cap B_k') \cap \lambda(G)')\tilde{u}_{k-1} \\ &= \pi_\alpha((A \cap B_k')^{\alpha^k}) = \pi_\alpha(A^{\alpha^{st}} \cap B_k'),\end{aligned}$$

and similarly $\widetilde{B} \cap \widetilde{B}_k' = \pi_\alpha(B^{\alpha^{st}} \cap B_k')$. Hence the core [11, 1.4.1] of $\widetilde{A} \supseteq \widetilde{B}$ associated to the tunnel constructed above is

$$\left(\bigvee_{k=0}^{\infty} \widetilde{A} \cap \widetilde{B}_k' \supseteq \bigvee_{k=0}^{\infty} \widetilde{B} \cap \widetilde{B}_k'\right) = (\pi_\alpha(A^{\alpha^{st}}) \supseteq \pi_\alpha(B^{\alpha^{st}})). \quad \square$$

6. Strong amenability

We can now prove the main result of this paper.

6.1. **Theorem.** *Assume that $A \supseteq B$ is extremal, and consider the following statements*:

(i) *$A \supseteq B$ is strongly amenable.*
(ii) *$\widetilde{A} \supseteq \widetilde{B}$ is strongly amenable.*

Then (ii) *implies* (i), *and if G is finite,* (i) *implies* (ii).

Note that the validity of (ii) automatically entails amenability of G.

Proof. Assume first that $\widetilde{A} \supseteq \widetilde{B}$ is strongly amenable. By (3.1) it is also extremal, so by (5.1) and [11, 5.3.1, 4.1.2], we have a scalar $\lambda \in \mathbb{C}$ such that

$$E_{(\bigvee_k \widetilde{A} \cap \widetilde{B}_k') \cap (\bigvee_k \widetilde{B} \cap \widetilde{B}_k')'}(e_0) = \lambda 1,$$

and by proving, as we do below, that

$$E_{(\bigvee_k A \cap B_k') \cap (\bigvee_k B \cap B_k')'}(e_0) = \lambda 1,$$

we get that $A \supseteq B$ is strongly amenable. By (5.1), we have

$$\widetilde{B} \cap \widetilde{B}_k' \subseteq \widetilde{A} \cap \widetilde{B}_k' \subseteq A \cap B_k',$$

whence

$$\begin{array}{ccc} (A \cap B_k') \cap (\widetilde{B} \cap \widetilde{B}_k')' & \subseteq & A \cap B_k' \\ \cup\!\mid & & \cup\!\mid \\ (\widetilde{A} \cap \widetilde{B}_k') \cap (\widetilde{B} \cap \widetilde{B}_k')' & \subseteq & \widetilde{A} \cap \widetilde{B}_k' \end{array}$$

is a commuting square. Since $e_0 \in \widetilde{A} \cap \widetilde{B}_k'$ and

$$(B \cap B_k')' \subseteq (B \cap \widetilde{B}_k')' = (\widetilde{B} \cap \widetilde{B}_k')',$$

we get

$$\begin{aligned} \|E_{A\cap B_k'\cap(B\cap B_k')'}(e_0) - \lambda 1\|_2 &= \|E_{A\cap B_k'\cap(B\cap B_k')'} E_{A\cap B_k'\cap(\widetilde{B}\cap\widetilde{B}_k')'}(e_0) - \lambda 1\|_2 \\ &\leq \|E_{A\cap B_k'\cap(\widetilde{B}\cap\widetilde{B}_k')'}(e_0) - \lambda 1\|_2 \\ &= \|E_{\widetilde{A}\cap\widetilde{B}_k'\cap(\widetilde{B}\cap\widetilde{B}_k')'}(e_0) - \lambda 1\|_2 \\ &\to 0 \qquad (k \to \infty). \end{aligned}$$

Next, assume (i). Since $\widetilde{A} \supseteq \widetilde{B}$ is extremal, by [11, 5.3.1] condition (ii) is equivalent to factoriality and extremality of the standard part of $\widetilde{A} \supseteq \widetilde{B}$, so by (5.2), all we have to prove is that $A^{\alpha^{st}} \supseteq B^{\alpha^{st}}$ is an inclusion of factors which is extremal. This is clear if $\alpha^{st} = 1$. Otherwise α^{st} is strongly outer by [12, 1.6], and in case $|G| < \infty$, we get (ii) thanks to (3.2). □

6.2. *Remark.* We expect equivalence of (i) and (ii) in the above theorem to hold even when $|G| = \infty$. As one observes immediately from our proof in the case $|G| < \infty$, it is necessary and sufficient to prove that (i) implies that $A^{\alpha^{st}} \supseteq B^{\alpha^{st}}$ is an inclusion of factors which is extremal. By [11, 4.1.2] it would then actually follow that $A^{\alpha^{st}} \supseteq B^{\alpha^{st}}$ is isomorphic to $\widetilde{A} \supseteq \widetilde{B}$. Combining this with [8, 4.2] we might thus get a method for constructing irreducible strongly amenable subfactors of infinite depth.

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DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF TOKYO, HONGO, TOKYO, 113, JAPAN

Current address: Matematisk Institut, Københavns Universitet, DK 2100 København Ø, Denmark

E-mail address: winslow@math.ku.dk